Automatic Multiplicative Sequences
(and automatic semigroups)

Jakub Konieczny

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Period-doubling sequence $a: \mathbb{N} \to \{+1, -1\}$

\[+\text{---------+--------+----------}+\ldots\]

There are several ways to define this sequence:

1. **Explicit formula:**
   \[a(n) = (-1)^{\nu_2(n)}\]
   \[\rightarrow \nu_2(n) = \nu \iff n = 2^\nu (2n_0 + 1)\]

2. **Recurrence:**
   \[a(2n + 1) = +1, \quad a(2n) = -a(n)\]

3. **Substitution:**
   \[+ \mapsto +-, \quad - \mapsto ++\]
   \[\rightarrow + \mapsto +- \mapsto +++ \mapsto +++++ \rightarrow ++++++ \ldots\]

4. **Automaton:**

   ![Automaton Diagram]

   This sequence also happens to be completely multiplicative:
   \[a(nm) = a(n)a(m)\].

**Question for today:** What other automatic multiplicative sequences are there?

\[\rightarrow \text{Multiplicative sequence: } a(nm) = a(n)a(m) \text{ if } n \perp m.\]
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3. Substitution: $+ \mapsto +1$, $- \mapsto 1$
   $\Rightarrow + \mapsto +1 \mapsto +1 \mapsto +1 \mapsto \ldots$

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![Diagram of an automaton]

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Automatic sequences

Some notation: We let $k$ denote the base in which we work.

- $\Sigma_k = \{0, 1, \ldots, k - 1\}$, the set of digits in base $k$;
- $\Sigma_k^*$ is the set of words over $\Sigma_k$, monoid with concatenation;
- for $n \in \mathbb{N}_0$, $(n)_k \in \Sigma_k^*$ is the base-$k$ expansion of $n$;
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A finite $k$-automaton consists of:

- a finite set of states $S$ with a distinguished initial state $s_0$;
- a transition function $\delta: S \times \Sigma_k \to S$;
- an output function $\tau: S \to \mathbb{C}$.

Computing the sequence:

- Extend $\delta$ to a map $S \times \Sigma_k^*$ with $\delta(s, uv) = \delta(\delta(s, u), v)$;
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Motivation

Why are we interested in classification of automatic multiplicative sequences?

- Automatic sequences give rise to one of the weakest notions of computability. Hence, for each class $C$ of sequences it is natural to ask which sequences in $C$ are automatic.

- Correlations of $k$-automatic and multiplicative sequences have been studied in the context of Sarnak conjecture. The question of equality appears as a natural “extreme” case.

- Algebraic power series $\sum_{n=0}^{\infty} a(n)X^n$, where $(a(n))_{n=0}^{\infty}$ is a multiplicative sequence, have been completely classified. Our problem is closely related to the finite field analogue.
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Möbius function $\mu : \mathbb{N}_0 \to \{+1, 0, -1\}$ is the multiplicative sequence given by:

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\mu(n) = \begin{cases} 
(-1)^s & \text{if } n \text{ is the product of } s \text{ different primes,} \\
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\end{cases}
$$

Pseudorandomness principle: “$\mu$ looks random, except for the ways it’s obviously not”.

Conjecture (Sarnak (2012))

The Möbius function is orthogonal to every deterministic sequence $(a(n))_{n=0}^{\infty}$:

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\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} a(n) \mu(n) = 0.
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$\rightarrow$ deterministic $=$ produced by a zero-entropy topological dynamical system

Example: automatic sequences are deterministic. $\rightarrow$ linear subword complexity

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### Theorem (Müllner (2016), special cases by various authors)

*Automatic sequences fulfill the Sarnak conjecture.*
Motivation: correlations of automatic and multiplicative sequences

Question: Which multiplicative sequences $\nu$, other than $\mu$ are orthogonal to all (nice) automatic sequences?

Minimal assumptions: Since periodic sequences are automatic, $\nu$ needs to be aperiodic, i.e. $\frac{1}{N} \sum_{n=0}^{N-1} \nu(An + B) \to 0$ for each $A \in \mathbb{N}$, $B \in \mathbb{N}_0$.

Theorem (Lemańczyk & Müllner (2018))

Let $(a(n))_{n=0}^{\infty}$ be a primitive automatic sequence and let $(\nu(n))_{n=0}^{\infty}$ be a bounded, aperiodic multiplicative sequence. Then $\frac{1}{N} \sum_{n=0}^{N-1} a(n)\nu(n) \to 0$ as $N \to \infty$.

$\longrightarrow$ primitive $\simeq$ behaves the same way on all long intervals
$\longrightarrow$ e.g. $n \mapsto (-1)^{\lfloor \log_k n \rfloor} = (-1)^{\text{length of } (n)_k}$ is not primitive

Opposite extreme: If $a = \nu$ is both automatic and multiplicative sequence then the correlation $\frac{1}{N} \sum_{n=0}^{N-1} \bar{a}(n)\nu(n) = \frac{1}{N} \sum_{n=0}^{N-1} |a(n)|^2$ is “as large as possible”.
$\longrightarrow$ new interesting phenomena, e.g. the sparse case
Motivation: correlations of automatic and multiplicative sequences

**Question:** Which multiplicative sequences \( \nu \), other than \( \mu \) are orthogonal to all (nice) automatic sequences?

**Minimal assumptions:** Since periodic sequences are automatic, \( \nu \) needs to be *aperiodic*, i.e. \( \frac{1}{N} \sum_{n=0}^{N-1} \nu(An + B) \to 0 \) for each \( A \in \mathbb{N}, B \in \mathbb{N}_0 \).

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**Theorem (Lemańczyk & Müllner (2018))**

Let \( (a(n))_{n=0}^\infty \) be a *primitive* automatic sequence and let \( (\nu(n))_{n=0}^\infty \) be a bounded, aperiodic multiplicative sequence. Then \( \frac{1}{N} \sum_{n=0}^{N-1} a(n)\nu(n) \to 0 \) as \( N \to \infty \).

\[ \longrightarrow \text{primitive } \sim \text{ behaves the same way on all long intervals} \]
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Motivation: correlations of automatic and multiplicative sequences

**Question:** Which multiplicative sequences \( \nu \), other than \( \mu \) are orthogonal to all (nice) automatic sequences?

**Minimal assumptions:** Since periodic sequences are automatic, \( \nu \) needs to be aperiodic, i.e.
\[
\frac{1}{N} \sum_{n=0}^{N-1} \nu(An + B) \to 0 \text{ for each } A \in \mathbb{N}, B \in \mathbb{N}_0.
\]

**Theorem (Lemańczyk & Müllner (2018))**

Let \((a(n))_{n=0}^{\infty}\) be a primitive automatic sequence and let \((\nu(n))_{n=0}^{\infty}\) be a bounded, aperiodic multiplicative sequence. Then
\[
\frac{1}{N} \sum_{n=0}^{N-1} a(n)\nu(n) \to 0 \text{ as } N \to \infty.
\]

\(\rightarrow\) primitive \(\sim\) behaves the same way on all long intervals
\(\rightarrow\) e.g. \(n \mapsto (-1)^{\lfloor \log_k n \rfloor} = (-1)^{\text{length of } (n)_k}\) is not primitive

**Opposite extreme:** If \(a = \nu\) is both automatic and multiplicative sequence then
the correlation
\[
\frac{1}{N} \sum_{n=0}^{N-1} \bar{a}(n)\nu(n) = \frac{1}{N} \sum_{n=0}^{N-1} |a(n)|^2
\]
is “as large as possible”. 
\(\rightarrow\) new interesting phenomena, e.g. the sparse case
Motivation — algebraic formal power series

Let \( K \) be a field, let \( a : \mathbb{N}_0 \to K \) be a sequence, and let \( F \) be the corresponding series:

\[
F(X) = \sum_{n=0}^{\infty} a(n)X^n \in K[[X]].
\]

Question

For which multiplicative sequences \((a(n))_{n=0}^{\infty}\) is \( F(X) \) algebraic over \( K(X) \)?

Theorem (Bézivin (1995), Bell, Bruin & Coons (2012))

If \( \text{char } K = 0 \), \((a(n))_n\) is multiplicative and \( F(X) \) is algebraic over \( K(X) \) then either

- \( a(n) = n^c \chi(n) \) for some \( c \in \mathbb{N}_0 \) and periodic \( \chi : \mathbb{N}_0 \to K \); or
- the sequence \( a \) is eventually zero.

Theorem (Christol (1979))

Let \( q \) be a prime power and let \( K = \mathbb{F}_q \). Then the following conditions are equivalent:

- the formal power series \( F(X) \) is algebraic over \( K(X) \);
- the sequence \((a(n))_{n=0}^{\infty}\) is \( q \)-automatic.
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Let $\mathbb{K}$ be a field, let $a: \mathbb{N}_0 \rightarrow \mathbb{K}$ be a sequence, and let $F$ be the corresponding series:

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**Question**

For which multiplicative sequences $(a(n))_{n=0}^{\infty}$ is $F(X)$ algebraic over $\mathbb{K}(X)$?

**Theorem (Bézivin (1995), Bell, Bruin & Coons (2012))**

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• S. Yazdani (2001) For many multiplicative sequences $f : \mathbb{N}_0 \to \mathbb{Z}$ of number-theoretic interest and $M \geq 2$, $n \mapsto f(n) \mod M$ is not automatic. (e.g. $f = \mu$ (Möbius), $\phi$ (totient), $\sigma_\ell$ (divisor sum), $\tau_\ell$ (divisor count), etc.) $\rightarrow$ explicit formulae, finiteness of $k$-kernels

• J.–C. Schlage–Puchta (2011) Each automatic, completely multiplicative, non-vanishing sequence is almost periodic ($d$-limit of periodic sequences). $\rightarrow$ mean values of multiplicative/automatic sequence

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- **J.-P. Allouche, L. Goldmakher (2018)** Study of “mock characters”: automatic completely multiplicative with extra assumptions. \( \rightarrow \) analogy to Dirichlet characters, “pretentious” number theory

- **O. Klurman & P. Kurlberg (2019) and Sh. Li (2020)** Each automatic, completely multiplicative sequence agrees with a Dirichlet character on large primes. \( \rightarrow \) Klurman–Kurlberg: heavy-duty number theory, e.g. work on Artin conjecture \( \rightarrow \) Li: combinatorics/automata

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Main result

Theorem (K., Lemańczyk, Müllner)

Let \( a : \mathbb{N}_0 \to \mathbb{C} \) be \( k \)-automatic, multiplicative, not eventually periodic. Then \( k \) is a power of a prime \( p \) and the sequence \( a \) takes the form

\[
a(n) = f(\nu_p(n)) \cdot g(n/p^{\nu_p(n)})
\]

(*)

where \( f \) and \( g \) are eventually periodic, \( f(0) = 1 \) and \( g \) is multiplicative.

\[\nu_p(n) = \max \{ \nu : p^\nu \mid n \}\]

Further comments:

- Conversely, any sequence of the form (*) is \( p \)-automatic and multiplicative.
- The representation in (*) is essentially unique.
- Any multiplicative sequence takes the form (*) where (necessarily):
  \[ f(n) = a(p^{\nu_p(n)}), \quad g(n) = a(n) \text{ if } p \nmid n, \quad g(p^{\alpha}) = 0 \text{ and } g \text{ is multiplicative.} \]
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Proof ideas: divide and conquer

Throughout, let \( a : \mathbb{N}_0 \to \mathbb{C} \) be \( k \)-automatic and multiplicative.

**Density dichotomy:** Consider the set

\[ P_0 = \{ p \in \mathcal{P} : \text{ there exists } \alpha \in \mathbb{N} \text{ with } a(p^\alpha) = 0 \} . \]

We will say that \( a \) is **dense** if \( |P_0| < \infty \) and **sparse** if \( |P_0| = \infty \).

**Lemma**

The following conditions are equivalent:

- the set \( P_0 \) is infinite,
- \( a(n) = 0 \) for almost all \( n \).

\[ \longrightarrow d \left( \{ n \in \mathbb{N} : a(n) \neq 0 \} \right) = 0 \]

**Sketch of a proof.**

- If \( |P_0| < \infty \) then \( a(n) \neq 0 \) for all \( n \) with \( n \equiv 1 \mod \prod_{p \in P_0} p \).
- Suppose \( |P_0| = \infty \) and let \( p \in P_0 \) be large, \( q = p^\alpha \), \( a(q) = 0 \). Then

\[ d_{\log} \left( \{ n : a(n) \neq 0 \} \right) = pq \cdot d_{\log} \left( \{ n : a(n) \neq 0, n \equiv q \mod pq \} \right) = 0 . \]
Proof ideas: divide and conquer

*Throughout, let \( a : \mathbb{N}_0 \rightarrow \mathbb{C} \) be \( k \)-automatic and multiplicative.*

**Density dichotomy:** Consider the set

\[
P_0 = \{ p \in \mathcal{P} : \text{ there exists } \alpha \in \mathbb{N} \text{ with } a(p^\alpha) = 0 \}.
\]

We will say that \( a \) is *dense* if \( |P_0| < \infty \) and *sparse* if \( |P_0| = \infty \).

**Lemma**

The following conditions are equivalent:

- the set \( P_0 \) is infinite,
- \( a(n) = 0 \) for almost all \( n \).

\[ \longrightarrow d(\{ n \in \mathbb{N} : a(n) \neq 0 \}) = 0 \]

**Sketch of a proof.**

- If \( |P_0| < \infty \) then \( a(n) \neq 0 \) for all \( n \) with \( n \equiv 1 \mod \prod_{p \in P_0} p \).
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The following conditions are equivalent:

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$$d_{\log} (\{ n : a(n) \neq 0 \}) = pq \cdot d_{\log} (\{ n : a(n) \neq 0, \ n \equiv q \mod pq \}) = 0.$$


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\[\square\]
Sparse case: arid sets

A basic $k$-arid set of rank $r$ takes the form

$$A = \left\{ [u_1 v_1^{\ell_1} u_2 v_2^{\ell_2} \ldots u_r v_r^{\ell_r} u_{r+1}]_k : \ell_1, \ell_2, \ldots, \ell_r \in \mathbb{N}_0 \right\}$$

where $u_1, \ldots, u_{r+1}, v_1, \ldots, v_{r+1} \in \Sigma_k^*$. A $k$-arid set is a union of basic $k$-arid sets. For example, the set of all $n \in \mathbb{N}_0$ whose base-10 expansion is increasing is 10-arid:

$$\left\{ [1^{\ell_1} 2^{\ell_2} \ldots 9^{\ell_9}]_k : \ell_1, \ell_2, \ldots, \ell_9 \in \mathbb{N}_0 \right\}.$$

**Proposition**

Let $a : \mathbb{N}_0 \to \mathbb{C}$ be $k$-automatic. Then one of the following holds:

- the set $\{n \in \mathbb{N}_0 : a(n) \neq 0\}$ is $k$-arid;
- there are $u, v_1 \neq v_2, w \in \Sigma_k^*$ with $|v_1| = |v_2|, a([uvw]_k) \neq 0$ for all $v \in \{v_1, v_2\}^*$.

**Consequence:** If $a$ is sparse then the set $\{n \in \mathbb{N}_0 : a(n) \neq 0\}$ is $k$-arid.

**Proof:** Given $u, v_1 \neq v_2, w \in \Sigma_k^*$, the set $\{[uvw]_k : v \in \{v_1, v_2\}^*\}$ intersects all residue classes modulo any power of a large prime.
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Sparse case: rank estimates

Story so far: The set \( Z := \{ n \in \mathbb{N}_0 : a(n) \neq 0 \} \) is \( k \)-arid.

Next goal: \( Z \) is a union of geometric progressions: \( Z = \bigcup_{i=1}^{s} \{ b_i k^{c_i \ell} : \ell \in \mathbb{N}_0 \} \).

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  \]
  where \( x_i \in \mathbb{Q}, a_i \in \mathbb{N}, \) all chosen so that the set above is contained in \( \mathbb{N}_0 \).

- Conversely, any set of the form (†) is \( k \)-arid of rank \( \leq r \).

- It follows that for \( k \)-arid sets \( A, B \) or ranks \( r, s \) respectively, there exists a \( k \)-arid set \( C \) of rank \( \leq rs + r + s - 1 \) such that \( A \cdot B \subset C \). Conversely: rank(\( C \)) \( \geq rs \).

- If \( a \) was completely multiplicative, then we would have \( Z \cdot Z \subset Z \). This is only possible if rank(\( Z \))^2 \( \leq \) rank(\( Z \)), so rank(\( Z \)) \( \leq 1 \).

- Without extra assumptions, \( Z \) is still closed under the operation \( (n, m) \mapsto nm/\gcd(n, m) \) which we can use to construct high rank \( k \)-arid set in \( Z \), leading to contradiction unless rank(\( Z \)) \( \leq 1 \).

- If rank(\( Z \)) = 1 and \( \{ xk^{a\ell} + y : \ell \in \mathbb{N}_0 \} \subset Z \) is one of its basic components, apply ideas above to show that \( y = 0 \).
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- If \( a \) was completely multiplicative, then we would have \( Z \cdot Z \subset Z \). This is only possible if \( \text{rank}(Z)^2 \leq \text{rank}(Z) \), so \( \text{rank}(Z) \leq 1 \).
- Without extra assumptions, \( Z \) is still closed under the operation \( (n, m) \mapsto nm / \gcd(n, m) \) which we can use to construct high rank \( k \)-arid set in \( Z \), leading to contradiction unless \( \text{rank}(Z) \leq 1 \).
- If \( \text{rank}(Z) = 1 \) and \( \{ x k^{a \ell} + y : \ell \in \mathbb{N}_0 \} \subset Z \) is one of its basic components, apply ideas above to show that \( y = 0 \).
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where $x_i \in \mathbb{Q}$, $a_i \in \mathbb{N}$, all chosen so that the set above is contained in $\mathbb{N}_0$.

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Sparse case: rank estimates

Story so far: The set $Z := \{n \in \mathbb{N}_0 : a(n) \neq 0\}$ is $k$-arid.

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- If \( \text{rank}(Z) = 1 \) and \( \{ xk^{a_\ell} + y : \ell \in \mathbb{N}_0 \} \subset Z \) is one of its basic components, apply ideas above to show that \( y = 0 \).
Sparse case: rank estimates

**Story so far:** The set $Z := \{ n \in \mathbb{N}_0 : a(n) \neq 0 \}$ is $k$-arid.

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- If rank($Z$) = 1 and $\{ x k^{a\ell} + y : \ell \in \mathbb{N}_0 \} \subset Z$ is one of its basic components, apply ideas above to show that $y = 0$. 
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- If $\text{rank}(Z) = 1$ and $\{ x k^{a\ell} + y : \ell \in \mathbb{N}_0 \} \subset Z$ is one of its basic components, apply ideas above to show that $y = 0$. 

Sparse case: end of the chase

Story so far: The set of non-zero places of \(a\) is a union of geometric progressions:

\[
\{ n \in \mathbb{N}_0 : a(n) \neq 0 \} = \bigcup_{i=1}^{s} \left\{ b_i k^{c_i \ell} : \ell \in \mathbb{N}_0 \right\}. 
\] (†)

Claim: The base \(k\) is a prime power (unless \(a\) is eventually zero).

Proof: Suppose \(p \mid k\), prime, \(k\) not a power of \(p\).

- There are \(\infty\) many \(\alpha \in \mathbb{N}_0\) with \(a(p^\alpha) \neq 0\).
- Each progression in (†) includes \(\leq 1\) power of \(p\);

\[\rightarrow \alpha = \nu_p(b_i) + c_i \nu_p(k) \ell \quad \rightarrow \text{iff} \quad b_i = p^{\alpha_i}\]

Back to main theorem: We may assume: \(k = p\) is prime; \(p \nmid b_i\) for all \(i\). Then

\[a(n) = f(\nu_p(n)) \cdot g(n/p^{\nu_p(n)}), \quad \text{where} \quad g := 1_{\{b_1, \ldots, b_s\}} \cdot a \quad \text{and} \quad f(\alpha) := a(p^\alpha). \quad (‡)\]

Since \(g\) is finitely supported, it is clearly eventually periodic.
Sparse case: end of the chase

**Story so far:** The set of non-zero places of $a$ is a union of geometric progressions:

$$\{n \in \mathbb{N}_0 : a(n) \neq 0\} = \bigcup_{i=1}^{s} \left\{ b_i k^{c_i \ell} : \ell \in \mathbb{N}_0 \right\}. \quad (\dagger)$$

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Dense case: large primes

Recall: We assume that $a$ is dense: $P_0$ is finite, $a(p^\alpha) \neq 0$ if $p$ is a large prime.

Proposition

There exists a threshold $p_*$ such that

$$a(p^\alpha) = a(p)^\alpha \neq 0 \text{ for all primes } p \geq p_* \text{ and all } \alpha \in \mathbb{N}.$$  

Once this is established, we follow in the footsteps of Klurman & Kurlberg (2019).

Theorem (Elliott & Kish (2017))

If $a : \mathbb{N}_0 \to \mathbb{C}$ is a completely multiplicative, $A, B, C, D \in \mathbb{N}$ with $AD - BC \neq 0$ and

$$a(An + B)/a(Cn + D) = \text{const.} \neq 0, \quad (n \in \mathbb{N}_0)$$

then there is a Dirichlet character $\chi$ such that $a(p) = \chi(p)$ for all large primes $p$.

Since $a$ is $k$-automatic, there are $\beta > \gamma$ with $a(k^\beta n + 1) = a(k^\gamma n + 1)$ for all $n \in \mathbb{N}_0$. Let $Q$ be the product of all the primes $p < p_*$. Then

$$a(k^\beta Qn + 1)/a(k^\gamma Qn + 1) = 1.$$  

By the theorem above, $a(p) = \chi(p)$ for large primes.
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Dense case: Restoring complete multiplicativity

Claim: There is $p_*$ such that $a(p^\alpha) = a(p)^\alpha$ for all primes $p \geq p_*$ and all $\alpha \in \mathbb{N}$.

- Define equivalence relation $\sim$ of $\mathbb{N}_0$, where $n \sim n'$ if the base-$k$ expansions $(n)_k, (n')_k$ “act in the same way” on the automaton that computes $a$:

$$\delta(\bullet, (n)_k) = \delta(\bullet, (n')_k) \quad \text{i.e.} \quad a(k^j + \ell x + k^j n + y) = a(k^j + \ell x + k^j n' + y)$$

for all $x, y, j, \ell \in \mathbb{N}_0$ with $y < k^j$ and $n, n' < k^\ell$. $\rightarrow$ no overlaps in sums above

- Because $a$ is automatic, the number of equivalence classes is finite: $|\mathbb{N}_0/\sim| < \infty$.

- Pick a large prime $p$ and $n, n'$ with $n \sim n'$, $pn \sim pn'$, $p \nmid n - n'$. $\rightarrow$ pidgeonhole

- Aiming to show (†) $a(pm) = a(p)a(m)$ for many $m \in \mathbb{N}$.

- Pick $m$ that “contains” $n$, that is, $m = k^j + \ell x + k^j n + y$. If $p \nmid m$ then (†) holds.

  - If $p \mid m$ (and $\ell$ is large) then $p \mid m' = k^j + \ell x + k^j n' + y$ so

    $$a(pm) = a\left(k^j + \ell (px) + k^j (pn) + (py)\right)$$

    $$= a\left(k^j + \ell (px) + k^j (pn') + (py)\right) = a(pm') = a(p)a(m') = a(p)a(m).$$

- Almost every $m$ “contains” $n$, so (†) holds for almost every $m \in \mathbb{N}_0$.

- In particular, there is $m$ with $a(m) \neq 0$, $\nu_p(m) = \alpha$, (†); so $a(p^{\alpha+1}) = a(p)a(p^{\alpha})$. 
Dense case: Restoring complete multiplicativity

Claim: There is $p_*$ such that $a(p^\alpha) = a(p)^\alpha$ for all primes $p \geq p_*$ and all $\alpha \in \mathbb{N}$.

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- Aiming to show $(\dagger) a(pm) = a(p)a(m)$ for many $m \in \mathbb{N}$.

- Pick $m$ that “contains” $n$, that is, $m = k^j+\ell x + k^j n + y$. If $p \nmid m$ then $(\dagger)$ holds.

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**Claim:** There is $p_*$ such that $a(p^\alpha) = a(p)\alpha$ for all primes $p \geq p_*$ and all $\alpha \in \mathbb{N}$.

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- Pick $m$ that “contains” $n$, that is, $m = k^j + \ell x + k^j n + y$. If $p \nmid m$ then (†) holds. If $p | m$ (and $\ell$ is large) then $p | m' = k^j + \ell x + k^j n' + y$ so

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**Claim:** There is \( p_* \) such that \( a(p^\alpha) = a(p)^\alpha \) for all primes \( p \geq p_* \) and all \( \alpha \in \mathbb{N} \).

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Dense case: Restoring complete multiplicativity

Claim: There is \( p_\ast \) such that \( a(p^\alpha) = a(p)^\alpha \) for all primes \( p \geq p_\ast \) and all \( \alpha \in \mathbb{N} \).

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Dense case: Restoring complete multiplicativity

Claim: There is $p_*$ such that $a(p^\alpha) = a(p)^\alpha$ for all primes $p \geq p_*$ and all $\alpha \in \mathbb{N}$.

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  for all $x, y, j, \ell \in \mathbb{N}_0$ with $y < k^j$ and $n, n' < k^\ell$.  \(\rightarrow\) no overlaps in sums above

- Because $a$ is automatic, the number of equivalence classes is finite: $|\mathbb{N}_0/\sim| < \infty$.

- Pick a large prime $p$ and $n, n'$ with $n \sim n'$, $pn \sim pn'$, $p \nmid n - n'$.  \(\rightarrow\) pigeonhole

- Aiming to show ($\dagger$) $a(pm) = a(p)a(m)$ for many $m \in \mathbb{N}$.

- Pick $m$ that “contains” $n$, that is, $m = k^j + \ell x + k^j n + y$. If $p \nmid m$ then ($\dagger$) holds. If $p \mid m$ (and $\ell$ is large) then $p \mid m' = k^j + \ell x + k^j n' + y$ so

  $$a(pm) = a\left(k^j + \ell (px) + k^j (pn) + (py)\right) = a\left(k^j + \ell (px) + k^j (pn') + (py)\right) = a(pm') = a(p)a(m') = a(p)a(m).$$

- Almost every $m$ “contains” $n$, so ($\dagger$) holds for almost every $m \in \mathbb{N}_0$.

- In particular, there is $m$ with $a(m) \neq 0$, $\nu_p(m) = \alpha$, ($\dagger$); so $a(p^{\alpha+1}) = a(p)a(p^\alpha)$. 
Dense case: Restoring complete multiplicativity

**Claim:** There is $p_*$ such that $a(p^\alpha) = a(p)\alpha$ for all primes $p \geq p_*$ and all $\alpha \in \mathbb{N}$.

- Define equivalence relation $\sim$ of $\mathbb{N}_0$, where $n \sim n'$ if the base-$k$ expansions $(n)_k, (n')_k$ “act in the same way” on the automaton that computes $a$:
  \[ \delta(\bullet, (n)_k) = \delta(\bullet, (n')_k) \quad \text{i.e.} \quad a(k^{j+\ell}x + k^j n + y) = a(k^{j+\ell}x + k^j n' + y) \]
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\[\text{16 / 23}\]
Proposition

For each prime $p \not| k$, the sequence $a(p^\alpha)$ ($\alpha \in \mathbb{N}$) is eventually periodic.

Proof outline:

- Pick $n \sim n'$ such that $\Delta := n' - n$ is divisible by all small primes.

- For given prime $p$ and $\alpha \in \mathbb{N}_0$, find a large prime $q$ such that: $a(q) = \chi(q) = 1$ and the base-$k$ expansion $(qp^\alpha)_k$ begins with $(n)_k$. $\rightarrow$ PNT in arithm. prog.

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- Nudging $q$ slightly, may ensure that $\chi(qp^{\alpha-\delta} + \Delta k^\beta/p^\delta) = \chi(p^{\alpha-\delta} + \Delta/p^\delta)$.

- It follows that $a(p^\alpha) = a(p^\delta)\chi(p^{\alpha-\delta} + \Delta/p^\delta)$ is a periodic function of $\alpha$. 
Dense case: small primes

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Dense case: prime divisors of $k$

**Fact:** If $k$ is a power of a prime $p$ then $a(p^\alpha)$ is eventually periodic.

$\rightarrow$ General fact about automatic sequences

**Proposition**

If $k$ is not a prime power, and $p \mid k$ then $a(p^\alpha)$ is eventually periodic.

$\rightarrow$ Same circle of ideas as on the previous slide

**Story so far:** We have a decomposition

$$a(n) = \left(\frac{\text{contribution from large primes}}{\text{contribution from small primes not dividing } k}\right) \times \left(\text{contributions from prime divisors of } k\right),$$

and we know that each component exhibits appropriate periodic behaviour: For each small prime $p$, the corresponding contribution is $p$-automatic.
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Dense case: Cobham-like theorems

**Theorem (Cobham (1969))**

Let \((a(n))_{n=0}^\infty\) be a sequence that is both \(k\)- and \(\ell\)-automatic. Then

- the bases \(k\) and \(\ell\) are rational powers of each other \(\rightarrow k\text{-auto.} = \ell\text{-auto.}\)
- the sequence \(a\) is eventually periodic \(\rightarrow m\text{-automatic for all } m\)

**Proposition**

Let \(a\) be a primitive \(k\)-automatic sequence, and for \(1 \leq i \leq s\) let \(b_i\) be primitive \(\ell_i\)-automatic sequences. Suppose further that there exists a map \(F\) such that

\[ a(n) = F(b_1(n), \ldots, b_s(n)). \]

and that \(k, \ell_1, \ldots, \ell_s\) are pairwise coprime. Then \(a\) is periodic.

But from the previous slide we have:

\[ a(n) \underbrace{\text{\(k\)-automatic}}_{\text{(periodic)}} \times \prod_{p \nmid k} (p\text{-automatic}) \times \prod_{q \mid k} (q\text{-automatic}) \]

When \(k\) is a prime power, it follows that \(a(n) = a(p^{\nu_p(n)}) \times (\text{periodic})\).

When \(k\) is not a prime power — story for another time.
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Automatic semigroups — motivation

Question

For a given \( k \in \mathbb{N}_{\geq 2} \), classify all \( E \subset \mathbb{N}_{0} \) that are simultaneously
- a \( k \)-automatic set \( \rightarrow 1_{E} \) is a \( k \)-automatic sequence
- a multiplicative semigroup \( \rightarrow n \cdot m \in E \) for all \( n,m \in E \)

Example: Let \( a : \mathbb{N}_{0} \to \mathbb{C} \) be an automatic completely multiplicative sequence. Then \( E := \{ n \in \mathbb{N}_{0} : a(n) = 1 \} \) is a semigroup. \( \rightarrow \) Morally: Classification of automatic semigroups implies classification of automatic completely multiplicative sequences.

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Automatic semigroups — motivation

Question

For a given $k \in \mathbb{N}_{\geq 2}$, classify all $E \subset \mathbb{N}_0$ that are simultaneously

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**Theorem (K. & Klurman (upcoming))**

Let $E \subset \mathbb{N}_0$ be a $k$-automatic semigroup, $k$ not a perfect power.

- If $\bar{d}(E) = 0$ then $E$ is a finite union of $k$-geometric progressions:

  $$E = \bigcup_{j=1}^{s} \{ k^{a_j \ell + c_j} : \ell \in \mathbb{N}_0 \} \text{ for some } s \in \mathbb{N}_0 \text{ and } a_j, c_j \in \mathbb{N}_0 \ (1 \leq j \leq s).$$

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**Remark:** Using the above as a black box, we can obtain some further “trivial” generalizations (e.g. if $k$ is a prime, we can drop the assumption that $n \perp k$ for all $n \in E.$) $\rightarrow$ “A statement is trivial if its proof requires less space than its statement.”
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**Proposition**

Let $E \subset \mathbb{N}_0$ be a $k$-automatic set. If $q \in \mathbb{N}$ has no small prime factors (i.e. $p \mid q \Rightarrow p > p_0(E)$) then $d_{\log}(E/q) = d_{\log}(E)$. $\rightarrow E/q = \{n \in \mathbb{N}_0 : qn \in E\}$

**Consequence:** Since $E$ is a semigroup, $E \subset E/q$ for all $q \in E$. If $q$ has no small prime factors, it follows that $d_{\log}(E/q \triangle E) = 0$. $\rightarrow X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$

**Wishful thinking:** Suppose that $E/q = E$ for all $q \in E$ with no small prime factors.

Consider the equivalence relation $\sim$ on $\mathbb{N}_0$, where:

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Then $\sim$ is compatible with multiplication and induces a quotient map $\mathbb{N}_0 \rightarrow \mathbb{N}_0/\sim$.

**Key observation:** $\mathbb{N}_0/\sim$ is not just any semigroup: it is a finite abelian 0-group (i.e., multiplicative group with added zero element). We can use previous results on multiplicative sequences to completely describe the map $\mathbb{N}_0 \rightarrow \mathbb{N}_0/\sim$ (cf. classification of finite abelian groups). Once we know $\sim$, we can reconstruct $E$. 
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Thank you for your attention!